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CONTINUOUS SPEED CHANGES FOR FLOWS

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ABSTRACT
Given a smooth flow on a compact manifold, it is shown that every measure theoretic isomorphism class that can be obtained by a measurable time change can already be obtained by a continuous time change with a derivative at every point.

1. Introduction

In recent years a lot of work has been done on the question of when two measure preserving flows are isomorphic. About a year ago, Feldman initiated a parallel theory [1] and [10] which treats the problem of determining which flows can be obtained from a given flow by changing the speed (but not the direction) of the flow along its orbits. (An equivalent formulation is: when can we find a 1-1 invertible measurable mapping taking orbits onto orbits, or which flows have the same orbit structure.) In particular, the theory characterizes the flows obtained by changing the speed in the Bernoulli flow.

All of this is done in the framework of measurable flows, but many of the examples which originally motivated the theory were smooth flows on manifolds. Thus, if we start with a smooth flow on a manifold, it is reasonable to ask what happens if we make a smooth speed change.

Our theorem says that any flow that can be obtained by a measurable speed change can be obtained by changing the speed so that the new speed is continuous and even has a differential at every point, but not necessarily a bounded one.

An example of special interest is the geodesic flow on a surface of negative curvature. This is known to be isomorphic to the Bernoulli flow [6] and we can

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thus characterize the flows that can be obtained by a measurable and hence continuous speed change. For example, we show in §4 that we can get uncountably many non-isomorphic K flows of the same entropy. On the other hand, if the speed change is smooth enough the flow must be the Bernoulli flow (because it is Anosov). (The minimal smoothness condition needed to force Bernoulliness is not known.)

Our result also has some bearing on the problem of whether every transformation or flow of finite entropy has a smooth model. In this direction Lind and Thouvenot have shown that any transformation of finite entropy is isomorphic to a homeomorphism of the 2-torus that preserves Lebesgue measure, while Katok recently constructed a smooth K-automorphism that is not Bernoulli or even loose Bernoulli.

2. Measurable flows and time changes

Let $(\Omega, \mathcal{B}, \mu)$ be a probability measure space. A *measurable flow* S_t, $t \in \mathbf{R}$ ($\mathbf{R}$ the real numbers) is a group of measurable transformations

$$S_t : \Omega \to \Omega$$

such that $S_{t+s} = S_t \circ S_s$, and the mapping

$$S : R \times \Omega \to \Omega$$

defined by $S(t, \omega)$ is measurable with respect to the product measurable structure of $\mathbf{R} \times \Omega$ ($\mathbf{R}$ equipped with the Borel structure). The flow is measure preserving if for all $t \in R$, S_t is a measure preserving transformation. Let $v(\omega) > 0$ be a real valued positive integrable function on Ω. ($[v(\omega)]^{-1}$ will be the new speed.) A new flow, which will be denoted by S_t^v, is obtained in the following way. Let

$$\bar{u}(\omega, d) = \int_0^d v(S_s\omega)ds$$

(if S_s moves at unit speed, $\bar{u}(\omega, d)$ will be the time it takes the new flow to move d units starting from ω). For fixed ω, $\bar{u}(\omega, d)$ is a strictly monotone function of d. Let $u(\omega, t)$ be the inverse function of $\bar{u}(\omega, t)$, i.e.,

$$\int_0^{u(\omega,t)} v(S_s\omega)\, ds = t.$$

($u(\omega, t)$ is the distance moved in t units of time.) Define the new flow by

$$S_t^v(\omega) = S_{u(\omega,t)}(\omega).$$

If S_t preserves the measure μ then S_t^v preserves an equivalent measure $\hat{\mu}$ which is given by $d\hat{\mu} = vd\mu$.

Two flows which are defined on the same space and which share a common cross-section and induce the same transformation on that cross-section can be obtained each from the other by such a speed change.

3. Speed changes with a derivative at every point

Let M be a compact $C^{(1)}$ manifold and S_t a $C^{(1)}$ flow on M which preserves a regular Borel measure of M. Consider all the flows S_t^v which are obtained from S_t by a measurable speed change, i.e., $v(\omega)$ is measurable. We show that every such flow is isomorphic to one which is obtained by a speed change which has a derivative at every point.

To this end we need the following:

LEMMA 1. *Let S_t be a smooth ergodic flow on a $C^{(1)}$ compact manifold M, preserving a Borel regular measure μ. Let $v(x) > 0$ be an integrable function on M. Let C be a closed subset of M of the form*

$$C = \bigcup_{0 \leq t \leq \rho} S_t A$$

where A is a smooth cross-section of S_t such that the return time to A is greater than ρ.

Then given $\varepsilon > 0$ we can find $\bar{v}(x) > 0$ integrable such that

(i) *$\bar{v}(x)$ is smooth on $M - C$.*

(ii) $|v(x) - \bar{v}(x)| < \varepsilon$ *(L_∞ norm).*

(iii) *There exists an isomorphism $\phi(x)$ between the flows S^v and $S^{\bar{v}}$ and $\mu\{\|x - \phi(x)\| < \varepsilon\} > 1 - \varepsilon$, where $\|x - y\|$ is a Riemannian distance between the point $x, y \in M$.*

PROOF. Let ν be a measure on A such that μ is represented as a product measure of ν and λ the Lebesgue measure on the part of $A \times R$ under the return function $f(x_0), x_0 \in A$, given by the flow S_t.

Approximate $v(x)$ by a smooth functions $\bar{v}$ in such a way that

$$\int |v(x) - \bar{v}(x)| d\mu(x) < \delta.$$

The choice of δ will be indicated later.

Consider the function

$$g(x_0) = \int_0^{f(x_0)} |v(S_t x_0) - \bar{v}(S_t x_0)| dt.$$

Using Fubini's theorem we can conclude that

$$\int_A g(x_0)d\nu < \delta.$$

Denote by T the ν-preserving transformation on the cross-section ν induced by the flow S_t.

Let $\gamma > 0$ be a fixed number $\gamma > \delta$. For $x_0 \in A$ let (k, l) be a pair of integers $k \leqq 0 \leqq l$ such that the following inequalities hold:

$$\frac{1}{l-k+1}\sum_{i=k}^{l} g(T^i x_0) < \gamma$$

and

$$\frac{1}{l'-k+1}\sum_{i=k}^{l'} g(T^i x_0) \geqq \gamma \qquad \text{for } k \leqq l' \leqq l.$$

For almost every point x_0, by the ergodic theorem, there is a pair $(k(x_0), l(x_0))$ such that $k(x_0)$ is minimal. It is an easy combinatorial fact that the pair $(k(x_0), l(x_0))$ is unique.

Partition the set A into $A_{k,l}$ sets such that

$$A_{k,l} = \{x_0 \mid (k(x_0), l(x_0)) = (k, l)\}.$$

The sets $A_{k,l}$ form a complete partition of A. Consider the set

$$\bar{A} = \bigcup_{k=0}^{\infty} A_{0,k}.$$

$\bar{A}$ is a measurable cross-section of the flow S_t. Since $S_t^{\tilde{v}}$ has the same orbits as S_t, $\bar{A}$ is also a cross-section for $S_t^{\tilde{v}}$. We want to change $\tilde{v}$ only on C in such a way that the return to $\bar{A}$ under the new flow will be the same as under S_t^{v}.

A point $x_0 \in A_{0,k}$ traverses the set C, $k+1$ times before it returns to $\bar{A}$. Assume, without loss of generality, that $\| S_t(\omega) - S_s(\omega) \| \leqq |t-s|$. Then, we can change $\tilde{v}$ to obtain a function $\bar{v}$, in such a way that

$$|\bar{v} - \tilde{v}| < \gamma/\rho$$

To indicate the choice of δ and γ we use the *Maximal Ergodic Theorem* to assert that

$$\nu(\bar{A} - A_{0,0}) < \frac{1}{\gamma}\int_A g(x_0)d\nu(x_0) \leqq \frac{\delta}{\gamma}.$$

We choose $\gamma = \frac{1}{2}(\varepsilon \cdot \rho)$ and $\delta < \frac{1}{2}\varepsilon$ and so small that

$$\int_{\bar{A}-A_{0,0}} f(x_0)d\nu(x_0) < \varepsilon,$$

where $f(x_0)$ is the return time function under the flow S_t to the cross-section A.

Now, we define the isomorphism ϕ in the following way.

Let $x_0 \in \bar{A}$ and $x = S_t^v x_0$, $0 \leqq t \leqq g(t)$ where $g(t)$ is the return time under S_t^v to $\bar{A}$ (which is equal almost everywhere to the return time under $S_t^{\bar{v}}$). Put

$$\phi(x) = S_t^v(x_0).$$

It is obvious that (i) and (ii) hold by the choice of $\bar{\omega}$ and that (iii) is satisfied on the set

$$\hat{A} = \{x : x = S_t^v(x_0), x_0 \in A_{0,0}, 0 \leqq t < g(x_0)\}.$$

We are now in position to prove the main result.

THEOREM 1. *Let S_t be a smooth ergodic flow on a $C^{(1)}$ compact manifold M, preserving a regular Borel measure μ. Let $v(x) > 0, x \in M$ be an integrable function and S_t^v the flow obtained from S_t by the time change induced by v.*

Then there exists a function $\hat{v}(x) > 0$ integrable and with a derivative at every point such that the flow $S_t^{\hat{v}}$ is measure theoretically isomorphic to S_t^v.

PROOF. Let $\{C_n\}$, $n = 1, 2, \cdots$ be a fixed sequence of closed sets which are obtained by "thickening" a smooth cross-section A_n, i.e.,

$$A_n = \bigcup_{0 \leqq t \leqq \rho_n} S_t A_n,$$

and such that the return time to A_n under S_t is greater than ρ_n, C_n are disjoint and C_n tend to a single point x_∞. That is, for every sequence $x_n \in C_n$, $x_n \to x_\infty$. Such a sequence $\{C_n\}$ certainly exists because locally the flow S_t is like a flow of unit speed along one of the coordinates in an m-dimensional euclidean space.

Let δ_n denote the distance between C_n and C_{n+1}, i.e.,

$$\delta_n = \inf\{\| x_n - x_{n+1} \| : x_n \in C_n ; x_{n+1} \in C_{n+1}\}.$$

Fix a sequence $\varepsilon_n = \delta_n / 2^n > 0$.

We use Lemma 1 to define a sequence of integrable functions $v_n(x) > 0$, $n = 0, 1, \cdots$ and mapping $\phi_n : M \to M$, $n = 1, 2, \cdots$ in the following way.

Put $v_0 = v$, and if v_{n-1} has been defined use Lemma 1 to get $v_n = \bar{v}$ and $\phi_n = \phi$ with respect to C_n, ε_n and v_{n-1}. Also, notice that in choosing $\bar{v}$ of Lemma 1 we can assume that $\bar{v} = v_{n-1}$ outside C_{n-1} because v_{n-1} is smooth there.

Now, by the choice of ε_n, $\lim v_n(x)$ exists everywhere. Call it $\hat{v}$. Using the Borel–Cantelli lemma we assert that for almost all x, $\| \phi_n(x) - x \| \geqq \varepsilon_n$ only a finite number of times and therefore the mapping

$$\psi_n = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi$$

converge almost everywhere to a mapping $\hat{\psi}$.

The function $\hat{v}$ is certainly smooth for all points different from x_∞. At x_∞ the function is equal to v_1. So in order to check the existence of its derivative there one has to check the difference ratio along sequence $\{x_n\}$, $x_n \in C_n$. Therefore, the existence follows from the choice of ε_n.

Let us show that $\hat{\psi}$ is an isomorphism mapping between S^v and $S^{\hat{v}}$. Fix t_0 and let x be a point such that $\lim \psi_n(x)$ exists. We have to show that if $y = S^v_{t_0}x$ then it takes the time t_0 for the point $\hat{\psi}_x$ to get $\hat{\psi}_y$ under $S^{\hat{v}}$. Obviously it takes time t_0 for $\psi_n x$ to get to $\psi_n y$ under S^{v_n} because ψ_n is an isomorphism map between S^v and S^{v_n}. Now, by continuity the claim follows.

4. K-flows of finite entropy

We now define a construction which will yield a class KF of uncountably many non-isomorphic K-flows all of the same finite entropy. Every such flow will be loosely Bernoulli (i.e., obtainable by a speed change of the finite entropy Bernoulli flow).

The construction will be a variant of the one described in [8].

The generating partitions of each $S_t \in K$F will consist of four sets $(P_f, P_s, P_{2_1}, P_{e_2})$.

Instead of defining the flow formally through the construction of the continuous nth gadget, as in [8], we shall rather describe informally the type of continuous n-blocks that arise.

An n-block will begin with a P_f-interval, the length of X_f. X_f will have uniform distribution on the n^2 values $\{1/n, 2/n, \cdots, (n^2-1)/n, n\}$. The P_f-interval will be followed by an $(n-1)$-block, then by a P_s-interval of length $s(n)$, then by an $(n-1)$-block, then by a P_s-interval of length $2s(n)$, etc, $\cdots$. There will be 2^n $(n-1)$-blocks separated by P_s-intervals with increasing order of length. Or, depending on a fixed function $g(n)$, the P_s-intervals will appear in decreasing order. The last P_s-interval will be followed by a P_{e_1}-interval of length $n \cdot X_f$ so that the one name will determine X_f and then a P_{e_2}-interval of length $n^2 + n - n_f - X_f$ (so, that the length of an n-block is fixed).

The function $s(n)$ will be of the form

$$s(n) = c \cdot n^4$$

where the constant c will be chosen large enough as in [4].

THEOREM 2. *Every flow belonging to the class KF is a K-flow of finite entropy* *and is obtainable from the Bernoulli flow by a time change. And, there are uncountably many non-isomorphic flows in* KF.

PROOF. We omit the proof of the K-property because it is completely similar to the proofs in [8] and [4].

To show that the flows in KF are of finite entropy we prove that $P = (P_f, P_s, P_{e_1}, P_{e_2})$ is a generating partition under the transformation S_1 for $S_t \in K$F.

We have to show that the discrete S_1-P-name determines the continuous S_t-P-name. Now, the discrete S_1-P-name certainly determines the type of the continuous n-block (which depends only on X_f of the block). We have to show that for given $\varepsilon > 0$ the S_1-P-name determines the beginning of the continuous n-block up to ε with probability one.

Given an n-block it will be nested, with pronanility one, in infinitely many m-blocks, $m > n$, such that $X_f(m)$ of those blocks will be such that

$$X_f(m)\,(\text{mod}\,1) < \varepsilon.$$

Also, because $X_f(\text{mod}\,1)$ are non-arithmetic, with probability one, one of the n-blocks will have a starting time (mod 1) which is less than ε. This will determine the starting time of that block up to ε. This in turn will determine the starting time of all the n-blocks which are nested in this m-block up to ε (because their type is known).

Finally, to prove that S_t is obtained from a Bernoulli flow it is enough to show that there is a measurable cross-section of the flow such that the induced transformation on it is isomorphic to a Bernoulli shift, since the Bernoulli flow has a Bernouili cross-section [3].

Such a cross-section is obtained if we consider the base of the nth continuous gadget of the flow (which we did not define formally). Namely, it is the set of all points such that the time zero is the beginning of an n-block.

The proof of the fact that this cross-section is isomorphic to a Bernoulli shift can be derived as an application of the cutting and independent stacking method. Such a proof was given in [9] for the $G(n)$ gadgets of the Ornstein–Shields class of K-automorphisms.

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